The loop-degrees of Jacobi diagrams and Lie algebras

(Preliminary Version)

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Abstract

For weight systems derived from finite-dimensional Lie algebras with metrics, the following series of theorems are stated. If a weight system is derived from a nilpotent Lie algebra, then it vanishes Jacobi diagrams including some loops and vice versa. If it is derived from an extension of nilpotent one, then it vanishes diagrams with at least two loops. For weight systems derived from other types, there exist diagrams with sufficiently many loops which are not vanished. The relations to invariants of knot-like objects are discussed, too. Nilpotent Lie algebras induce the Milnor invariants of string links and Heisenberg Lie algebras do the inverse of the Alexander-Conway invariants of knots.

Chapter 1

Introduction

We start with backgrounds and motivations of our study.

1.1 History

Roughly speaking, a quantum invariant is a link invariant (a tangle invariant in general) with some parameters defined with a Lie algebra and its representation. As described below, quantum invariants are constructed in two ways, one is the construction by means of quantum groups (which are quantizations of Lie algebras), and the other is by use of ordinary Lie algebras.

The history of quantum invariants has started in the middle of '80s, when the Jones polynomial is discovered [10]. Jones has constructed an algebra in studying operator algebras, and defined a representation of the braid groups on it. The Jones polynomial of a link is a modified character of the action.

Although the Jones polynomial was derived from the theory of operator algebras in origin, it is related to many topics in mathematics and physics, for example representation theory of quantum groups, solvable lattice models in statistical physics, and quantum field theories. They contribute to the general framework of quantum invariants.

The braid groups also act on tensor representations of a quantum group, and thus a braid gives an intertwiner of the representations. It is an invariant of braids. In particular with the quantum group $U_q(sl_2)$, the invariant of braids and links coincides with the Jones polynomial of them. Various calculations and results obtained by replacing $U_q(sl_2)$ with other quantum groups are done.

One of the key that a quantum group derives an invariant of braids is that it gives a solution of the Yang-Baxter equation which appears in statistical physics. With an analogy between projection images of links and lattices, the Jones polynomial is constructed like as a partition function of a lattice model [11]. Moreover it leads us to the extension to invariants of tangles [31, 1], and thus quantum invariants have been defined without braid representation of links.

Witten [33] raised a proposal that the Jones polynomial is a partition function of 2-dimensional conformal field theories. He regarded a link as an orbit of moving particles in a plane and the Jones polynomial is obtained by Feynman's path integral.

Turaev [29] has given the categorical framework of quantum invariants. In his work, tangles are considered as morphisms of a certain monoidal category, and quantum invariants are functors from the category of tangle diagrams to the monoidal category of representations of quantum groups. Moreover, initial data are generalized to a braided Hopf algebra and its representation.

Bar-Natan and Kontsevich have reconstructed quantum invariants from the viewpoint of Vassiliev invariants. Vassiliev invariants are a large class of link invariants including all quantum invariants. It has been developed since '90 s.

A Vassiliev invariant is realized as a composite map $W \circ \hat{Z} : \{\text{links}\} \to \mathbb{C}[[h]]$. $\hat{Z}$ is a link invariant defined by Kontsevich [14], it takes values in the Hopf algebra of infinite formal sums of graphs (they are Jacobi diagrams, defined in Chapter 2). The Kontsevich invariant $\hat{Z}$ is universal for any Vassiliev invariants. Despite of the importance the properties of $\hat{Z}$ are not revealed in detail. One of the reasons is that it includes complicated elements called associators. There are partial results on the elements [3, 18]. Le and Murakami has shown in [19] a complete expression of an associator in which the multiple zeta values appear.

W is a function on the space of the graphs, it is called a weight system. Bar-Natan [2] has revealed the Hopf algebra structure of the space, and showed how to construct a weight system[1] $W_{L,t,\rho}$ from a Lie algebra L, its ad-invariant symmetric 2-tensor t, and a representation ρ of it. Le and Murakami [20] has proved the resulting invariant $W_{L,t,\rho} \circ \hat{Z}$ coincides with the quantum invariant constructed by Turaev's method if L is semisimple.

1.2 Problems and Results

It is a problem what is seen if we start with non semisimple Lie algebras. It is difficult to manipulate such algebras and their representations. Indeed, there are a few ways to do it. Moreover, it is not clear whether they have ad-invariant symmetric 2-tensors. Partial results are in [24].

However, it is worth consideration. One reason is that every Vassiliev invariant does not alway come from a semisimple Lie algebra [17], and another reason is the relation with the concept of loop-degrees.

Melvin and Morton have conjectured that for knots the inverse of the Alexander-Conway polynomial is the 'diagonal' part of the colored Jones polynomial [21]. After

[1] In the sequel, we omit the symbols t and ρ if they are clear.

it is proved[2] by Bar-Natan and Garoufalidis [5], a gradation, different from ordinary one, of Jacobi diagrams is defined and researched actively. The Kontsevich invariants for some objects (knots, links, braids, and more) are known to be grouplike, and hence the formal logarithms of them are presented as linear spans of connected Jacobi diagrams. The loop-degree of a connected Jacobi diagram D is the negative of the Euler characteristic of D. It defines a gradation of the space of connected Jacobi diagrams.

Jacobi diagrams having negative loop-degrees are related to (self-)linking numbers, more generally to the Milnor invariants [4, 9]. Furthermore the diagrams of loop-degree 0 are related to the Alexander-Conway polynomial of links, which indicates the Melvin-Morton conjecture. It can be defined as the quantum invariant derived from the Lie superalgebra $gl(1|1)$ [15]. Moreover the Kontsevich invariant of the unknot is written by only diagrams with loop-degree 0 [6, 28].

We consider weight systems derived from Lie algebras with the following properties. Fix an integer k and take a weight system w.

- If D is a connected Jacobi diagram of loop-degree $> k$, then $w(D) = 0$.
- There exists a connected Jacobi diagram D of loop-degree k such that $w(D) \neq 0$.

We call such a weight system loop-order k.

In the present paper the following theorems are proved. For a Lie algebra L with a metric, we denote by W_L the weight system constructed by L and its invariant tensor defined as the dual of the metric. Representations are not needed.

Theorem 1 *If L is nilpotent, then the loop-order of the weight system W_L are -1. Conversely, if the loop-order of a weight system W_L is negative, then L is nilpotent.*

Theorem 2 *If there exists a connected Jacobi diagram of loop-degree $k \geq 1$ which is not annihilated by W_L, then there exists a Jacobi diagram of loop-degree $> k$ with the same property.*

As a corollary, we can see that positive loop-orders are equivalent to the loop-order ∞ for weight systems derived from Lie algebras.

The proofs of theorems above are due to the fact that an endomorphism of a $\mathbb{C}$-vector space has at least one non-zero eigenvalue and an eigenvector associated with it. The improvement and the generalization, in particular for the case that coefficients are in $\mathbb{Q}$ or $\mathbb{Z}$, are problems.

By these theorems we state the main theorem.

[2] Rozansky has already 'proved' the conjecture then, by means of path integrals, which are not rigorous in mathematics [26].

Theorem 3 *If a weight system W_L is derived from a finite-dimensional Lie algebra, then its loop-order is any one of -1, 0, and ∞.*

It is still a problem that the existences of algebras deriving weight systems having distinguished loop-orders. In the present paper we also show some examples of them and research properties of their deriving weight systems. As a conclusion, we can see what kind of classes of algebras induce weight systems of fixed loop-orders as follows (or as pictured in Figure 1.1).

Theorem 4 *For any connected Jacobi diagram of loop-degree -1, there is a nilpotent Lie algebra such that the deriving weight system does not annihilate the diagram.*

It is proved in Chapter 4 by defining nilpotent Lie algebras $\mathfrak{g}(n)$ ($n \in \mathbb{N}$) with the required properties (Corollary 4.1.2). Moreover the derived quantum invariants are related to the Milnor invariants of string links (Theorem 7, in Chapter 4).

Some of solvable Lie algebras with metrics induce weight systems of loop-orders 0. Each of these algebras is known to be an extension of smaller one, extended by a central element and a derivation. The typical ones are Heisenberg Lie algebras.

Theorem 5 *Let L is a finite-dimensional solvable Lie algebra with a metric. If L is obtained by an extension of a nilpotent Lie algebra with a central element and a derivation*[3]*, and if L is not nilpotent, then the derived weight system is of loop-order* 0.

The derived quantum invariants have informations of the Alexander-Conway polynomial of knots (1-string links, precisely). It is described in Theorem 9, in Chapter 5.

The other types of algebras, including semisimple Lie algebras, are good examples for Theorem 2, although we do not deal with here.

1.3 Contents

The present paper is organized as follows.

Chapter 2 is devoted to preliminaries and general theories. We review the notion of Jacobi diagrams and weight systems. Weight systems are constructed from Lie algebras and some formulas are shown.

In Chapter 3, main results of this paper are proved. At first the structures of solvable Lie algebras with metrics are reviewed. Next we evaluate weight systems on wheel diagrams. It results in Theorem 1. Theorem 2 is proved by showing a diagram

[3] As we see in Chapter 5, algebras extended two times also induce loop-order 0 weight systems. However the third extensions do not have the properties.

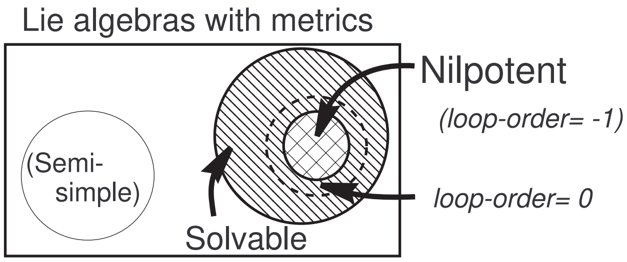

Figure 1.1: The classification of Lie algebras

of loop-degree > 1 not vanished. Theorem 3 and Theorem 5 are also proved here. Theorem 4 is proved in the next chapter.

In Chapter 4, nilpotent Lie algebras are dealt. As a proof of Theorem 4 we construct explicitly an algebra $\mathfrak{g}(n)$ and investigate the weight system derived from it. The Milnor invariants of string links are determined from the Kontsevich invariants by taking a quotient of the space of Jacobi diagrams. We see it is realized by evaluating $W_{\mathfrak{g}(n)}$ and we relate it to the Milnor invariants (Theorem 7).

We see in Chapter 5 weight systems derived from algebras which are solvable but not nilpotent. We deal with Heisenberg Lie algebras, as typical examples. Since they have triangular decompositions, we can imitate the semisimple cases. Resulting knot invariants are related to the inverse of Alexander-Conway polynomial(Theorem 9). The evaluations for vertical strands are also shown, which give a partial expression of the Drinfeld associator. Next, other types of solvable Lie algebras are described.

In the last chapter, some problems are raised.

Acknowledgement

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Chapter 2

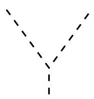
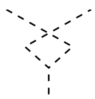
Preliminaries

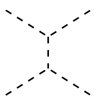
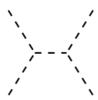
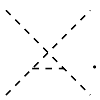
The contents of the chapter is elementary in the theory of Vassiliev invariants[1]. Throughout the paper the ground field is fixed to $\mathbb{C}$, the set of complex numbers, and we denote Kronecker's delta by δ_{ij}.

2.1 Jacobi diagrams

A Jacobi diagram is a graph such that each vertex of it is univalent or trivalent, and that each trivalent vertex is oriented. Its edges are often called chords, and univalent vertices are labeled by distinct symbols. In this paper, all Jacobi diagrams are drawn by dashed lines where each univalent vertex is marked. We also call a dashed graph a web.

Let us consider the linear space spanned by all Jacobi diagrams. The space $\mathcal{B}$ is the quotient of the linear space by the following relations. It is noticed that they are local relations. In other words, each of them is a relation of diagrams which are same except for the parts drawn explicitly.

AS:  $+$  $= 0.$

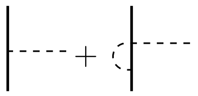
IHX:  $=$  $-$  .

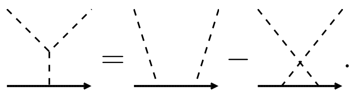
By the AS relation, we can abbreviate notions of orientations of trivalent vertices. The subspace of $\mathcal{B}$ generated by connected Jacobi diagrams is denoted by $\mathcal{P}$.

Let X be an oriented compact 1-manifold. A Jacobi diagram with support X is a pair of a Jacobi diagram and X such that all univalent vertices are attached with

[1]It is referred the monumental classical paper [2] and the modern book [25].

distinct points in X. Let us take the linear space of Jacobi diagrams with support X. Like as $\mathcal{B}$, we define the space $\mathcal{A}(X)$ as a quotient by four relations AS, AS', IHX, and STU, where relations AS' and STU are pictured below.

AS':  $= 0.$

STU: 

In particular, if X is k oriented segments, then $\mathcal{A}(X)$ is denoted by $\mathcal{A}(k)$.

For a Jacobi diagram D two types of degrees are defined. Let $v_3(D)$ be [resp. $v_1(D)$] the number of trivalent [resp. univalent] vertices in D. The ordinary degree $\deg(D)$ is the sum $\frac{1}{2}(v_3(D)+v_1(D))$ and the loop-degree is the difference $\frac{1}{2}(v_3(D)-v_1(D))$. It is noted that the relations above except for STU are homogenous with respect to both degrees, and that the loop-degree is the negative of the Euler number of the web of D.

Some examples are shown in Figure 2.1. A wheel diagram is defined as a diagram which has only one loop in it (depicted in the center). Wheel diagrams play important roles in this paper. Each trivalent vertex in a wheel diagram touches a univalent vertex and two trivalent ones. The wheel diagram with l univalent vertices are denoted by ω_l. The diagram in the center of Figure 2.1 is ω_6. The diagram in the left is an example of a tree graph. Tree graphs (including the diagram consisiting of a single chord) have loop-degrees -1. The diagram on the right is of loop-degree 1.

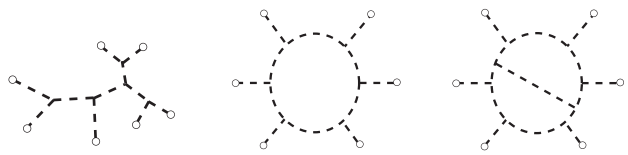


Figure 2.1: Examples of Jacobi diagrams

2.2 Weight systems coming from Lie algebras

Let L be a Lie algebra. An invariant metric of L is a non-degenerate bilinear form $(\cdot\,|\,\cdot)$ of L satisfying the relation $(x|[y,z]) = ([x,y]|z)$ for all $x,y,z \in L$. We will often omit the adjective 'invariant'.

Let us take a finite-dimensional Lie algebra L with a metric. We can take two bases $\{u_i\}$ and $\{v_j\}$ of L which satisfy $(u_i|v_j) = \delta_{ij}$. The dual of the metric, expressed as $t = \sum(u_i \otimes v_i + v_i \otimes u_i)$, is called an invariant 2-tensor of L. It has the following two properties.

- t is symmetric. Namely it is preserved by a permutation $u \otimes v \to v \otimes u$.
- t is invariant. In $U(L) \otimes U(L)$, where $U(L)$ is the universal enveloping algebra of L,
$$[t, \Delta(x)] = [t, 1 \otimes x + x \otimes 1] = 0$$
holds for $x \in L$.

The bracket product of L induces a 3-invariant tensor of L^*. We use symbols like u^i [resp. v^i], which is a basis vector of L^* determined by $u^i = (v_i|\cdot)$ [resp. $v^i = (u_i|\cdot)$]. Let us define the numbers $\{f_{ijk}\}$ by $[u_i, u_j] = \sum_k f_{ijk} v_k$, and $F = \sum f_{ijk} u^i \otimes u^j \otimes u^k$. It is noticed that F changes to $(\mathrm{sgn}\sigma)F$ by a permutation σ of indices, which is regarded as in S_3, the symmetric group of degree 3.

With data mentioned above, we can associate a tensor $W_L(D)$ with a Jacobi diagram $D \in \mathcal{B}$. First we assign the 3-tensor F [resp. an invariant 2-tensor t] to each trivalent vertex [resp. edge] in D. Next we contract all of the tensors, that is u^i of F and u_i (or v_i) of t should be evaluated according to connections of vertices and edges. The result is $W_L(D)$ (Figure 2.2 is an example). If a Jacobi diagram D has k univalent vertices, then $W_L(D)$ takes the value in $L^{\otimes k}$. We call the map $\mathcal{B} \to S(L)$ the weight system derived from L, where $S(L)$ is the symmetrical algebra of L.

The weight system W_L preserves two relations AS and IHX, and thus we have the following proposition.

Proposition 2.2.1 *W_L is a well-defined map from $\mathcal{B}$ to $S(L)$.*

Let us fix the support X and fix a representation R of L. We can also define weight systems on $\mathcal{A}(X)$, denoted by $W_{L,R}$, like as on $\mathcal{B}$. Additionally, we assign R to a piece of supports to which a univalent vertex is connected (Figure 2.3). $W_{L,R}$ preserves the relations AS, AS', IHX, and STU, the defining relations of $\mathcal{A}(X)$.

Proposition 2.2.2 *Assume that X has $2k$ boundary points ($k > 0$). Then $W_{L,R}$ is a well-defined map from $\mathcal{A}(X)$ to $\mathrm{End}(V_R^{\otimes k})$ where V_R is the representation space associated with R.*

A Jacobi diagram in $\mathcal{B}$ can be considered as a part of a diagram with a support. In this paper, we do not discuss the differences depending on the choice of representations. Thus, we denote weight systems for both diagrams with and without supports by W_L.

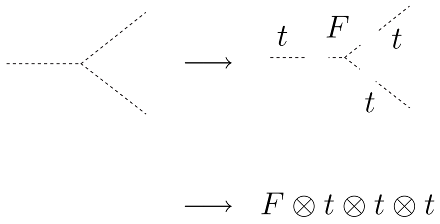


Figure 2.2: The evaluation of a weight system

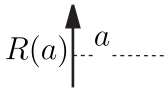


Figure 2.3: The evaluation around a support

Some formulas about evaluating W_L are shown. The first lemma assures us the evaluation is expressed by nested bracket products (Figure 2.2).

Lemma 2.2.3 *The part of a Jacobi diagram consisting of an edge and a trivalent vertex represents the bracket product of L. Precisely, contracting the diagram and two edges, to which the elements u_i and u_j are assigned, results in the assignment of $[u_i, u_j]$ to the open edge.*

PROOF. The evaluation is

$$\begin{aligned}
&(\sum f_{ijk} u^i \otimes u^j \otimes u^k) \otimes \sum u_l \otimes v_l \\
\to\ & \sum f_{ijk} u^i \otimes u^j \otimes u^k(u_l) \otimes v_l \\
=\ & \sum f_{ijk} u^i \otimes u^j \otimes v_k
\end{aligned}$$

and the result means the bracket product. $\square$

Lemma 2.2.4 *Let L and L' be finite-dimensional Lie algebras with metrics. For $D \in \mathcal{P}$, the following equality holds*

$$W_{L \oplus L'}(D) = W_L(D) + W_{L'}(D).$$

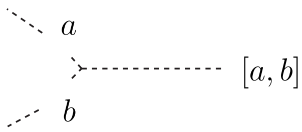
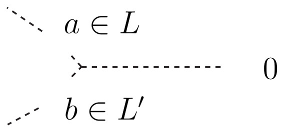


Figure 2.4: The graphical expression of Lemma 2.2.3

PROOF. The commutativity of L and L' in $L \oplus L'$ implies the vanishing of the contraction around a vertex, like as follows.



Hence, any term in the evaluation of $W_{L\oplus L'}(D)$ consists of elements in either L or L'. □

The following lemma is indicated by the fact that connecting a chord into a vertex and removing it can be regarded as taking its dual $a^* \leftrightarrow a$.

Lemma 2.2.5 *Let D be a Jacobi diagram with at least 2 univalent vertices. Let us consider the modification of D, obtained by removing a chord connected to a univalent vertex and by marking another univalent vertex. Then it defines an endomorphism of $S(L) \otimes L$.*

The lemma is clear by Figure 2.5. If an element $g \otimes v \in S(L) \otimes L$ is plugged in the lowest vertex, then another element $g \otimes g_1 \otimes g_2 \otimes g_3 \otimes v' \in S(L) \otimes L$ is obtained.

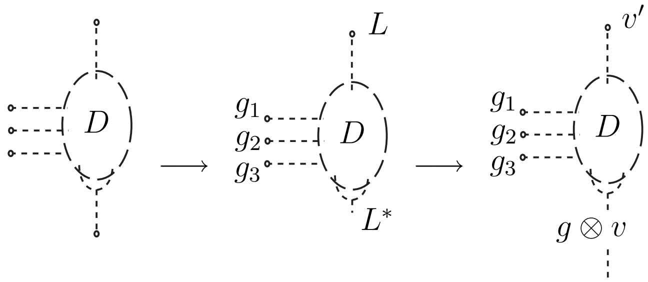


Figure 2.5: A diagram with some univalent vertices induce an endomorphism

Lemma 2.2.6 *Let D be a Jacobi diagram with a non-negative loop-degree. Fix an edge e of D which is a part of a loop. Then any term in $W_L(D)$ can be expressed as a trace function of an endomorphism $S(L) \otimes L \to S(L) \otimes L$.*

PROOF. Assume that labels on univalent vertices of D are fixed. Unplugging e defines an endomorphism[2] ϕ_D of $S(L) \otimes L$ as in the last lemma. It can be written as $\phi_D = \sum a_{ij} u_j \otimes u^i$ where each a_{ij} is in $S(L)$. Plugging e again induces the contraction as follows

$$\begin{aligned} \sum a_{ij} u_j \otimes u^i &\to \sum a_{ij} u^i(u_j) \\ &= \sum a_{ii} \\ &= \operatorname{Trace} \phi_D \end{aligned}$$

and the lemma is obtained. □

Example. Let D_1 be a Jacobi diagram of loop-degree 1 without univalent vertices. It is decomposed into two brackets and a chord (Figure 2.6), and is mapped by W_L for a Lie algebra L like as

$$W_L(D_1) = \operatorname{Trace}(\sum_i [u_i, [v_i, \cdot\,]]).$$

Similarly, it is remarked that, in evaluating on a diagram with a circle as a support, the result is presented as a trace of the fixed representation. For the diagram D_2 in Figure 2.7,

$$W_L(D_2) = \operatorname{Trace}(\sum_i R(u_i) R(u_i)).$$

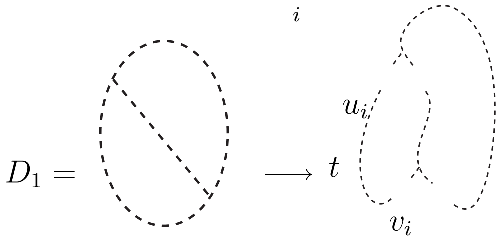


Figure 2.6: An example of the evaluation

The determination and researches, in particular the relations to topologies of knot-like objects, of the kernel of W_L for a Lie algebra L is an important problem and is related to the contents of this paper. If L has a central element c such that $(c|c) \neq 0$, then we can deform the metric and invariant tensor t is changed

[2] Of course it depends on labels on univalent vertices, though we omit them in this proof.

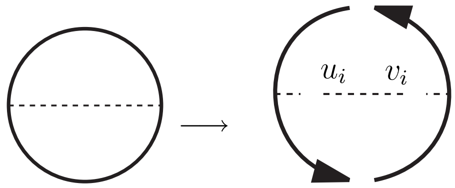
$D_2 =$ 

Figure 2.7: An example of the evaluation (with a support)

to $t' = t + \lambda c \otimes c$ where λ is a parameter. By an appropriate choice of λ, $W_{L,t'}$ vanishes diagram having the part like  [5]. This vanishing is often called '1T-relation' or 'framing-independence relation' on the level of Jacobi diagrams. $W_{L,t'}$ defines framing independent link invariants.

In Chapter 4, we deal with weight systems of loop-order -1. It vanishes all Jacobi diagrams with some loops. It corresponds to the fact such weight systems do not pick informations about self-linkings.

Chapter 3

Proofs of theorems

In this chapter, theorems except for Theorem 4 are proved. Before proving main theorems of this paper, we review some structures of solvable Lie algebras with metrics. The structures and some evaluations in the algebras induce Theorem 1, Theorem 5. Theorem 2 is proved without detailed properties of Lie algebras. Theorem 3 is a natural consequence of these theorems. Theorem 4 is dealt in Chapter 4

In this chapter all Lie algebras are assumed to be finite dimensional and to have metrics. We also assume that every algebra is not a direct sum of two Lie algebras.

3.1 The structure of solvable Lie algebras with metrics

At first, an extension of Lie algebras are introduced. Let L_0 be a Lie algebra with the bracket $[\,\cdot\,,\,\cdot\,]_0$ and a metric $(\,\cdot\,|\,\cdot\,)_0$. Let us consider a derivation $\bar{d}$ of L_0 such that $(\bar{d}(x)|y) + (x|\bar{d}(y)) = 0$ for $x, y \in L_0$. Then the vector space $L_0 \oplus \langle z, d\rangle$ has a structure of a metrized Lie algebra with the following bracket products $[\,\cdot\,,\,\cdot\,]$ and metrics $(\,\cdot\,|\,\cdot\,)$.

$$\begin{gathered}
[x, y] = [x, y]_0 + (\bar{d}(x)|y)z, \\
[z, x] = [z, d] = 0, \quad [d, x] = \bar{d}(x), \\
(x|y) = (x|y)_0, \\
(c|c) = (d|d) = (c|x) = (d|x) = 0, \\
(c|d) = 1.
\end{gathered}$$

Here x and y are in L_0. We often abuse the symbol d for the associated derivation $\bar{d}$.

Proposition 3.1.1 ([12, Exercise 2.11]) *Let L be a solvable n-dimensional Lie algebra with a metric. Then there exists a solvable Lie algebra L_0 such that*

- *L_0 is $(n-1)$-dimensional, and L is an orthogonal direct sum of L_0 and a 1-dimensional Lie algebra $\langle c\rangle$ with $(c|c)=1$ or,*

- *L_0 is $(n-2)$-dimensional, and L is an extension with two additional generators described above.*

The proposition says that any solvable Lie algebra is constructed with repeated extensions. Let us fix a non-negative integer k and denote by $\mathcal{C}^{(k)}$ a commutative Lie algebra generated by $c_1^{(k)}$, $c_2^{(k)}$, $\cdots$, $c_{l_k}^{(k)}$. The algebra $\mathcal{C}^{(k)}$ is metrized by $(c_i^{(k)}|c_j^{(k)})=\delta_{ij}$. For a solvable Lie algebra L, there exists an integer n and a sequence of solvable Lie algebras $\mathcal{S}_i=\mathcal{S}_{i,\bar{d}_i}$ $(0\leq i\leq n)$ such that $\mathcal{S}_0=\{0\}$ and $\mathcal{S}_n=L$ and $\mathcal{S}_{i+1}$ is an extension of the direct sum[1] $\mathcal{S}_i\oplus\mathcal{C}^{(i)}$ with a derivation $\bar{d}_{i+1}$ and a central element z_{i+1}.

Nilpotent Lie algebras are also solvable, and they are constructed with the same method. Moreover there is an explicit characterization.

Proposition 3.1.2 *A solvable finite dimensional Lie algebra with a metric is nilpotent if and only if all $\bar{d}_i$'s are nilpotent.*

Proof. Let us take a sequence of solvable algebras $\{S_i\}_{i=0}^n$ defined above. If $\mathcal{S}_n$ is nilpotent, then we can easily see the nilpotentness of all the derivations.

We prove that $\mathcal{S}_n$ is nilpotent by induction, under the assumption that all $\bar{d}_i$'s are nilpotent. At first, it is clear that $\mathcal{S}_1$ is nilpotent. Suppose that $\mathcal{S}_i$ is nilpotent for $i<m$. Let us consider the descending central series of $\mathcal{S}_m$. The k-th algebra $\underbrace{[\mathcal{S}_m,[\mathcal{S}_m,\cdots,[\mathcal{S}_m,\mathcal{S}_m]]]}_{k}$ in the series consists of nested brackets of $\mathcal{S}_{m-1}$ and its images by $\bar{d}_m$. The nilpotentness of $\mathcal{S}_{m-1}$ and of $\bar{d}_m$ implies that those brackets and images are vanished for large k. Hence the k-th algebra vanishes and $\mathcal{S}_m$ is nilpotent. $\square$

3.2 Proofs of theorems

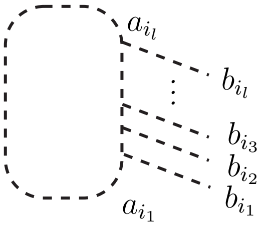
At first, we see the evaluations for wheel diagrams ω_l.

Lemma 3.2.1 *For a Lie algebra L, $W_L(\omega_l)$ is expressed as*

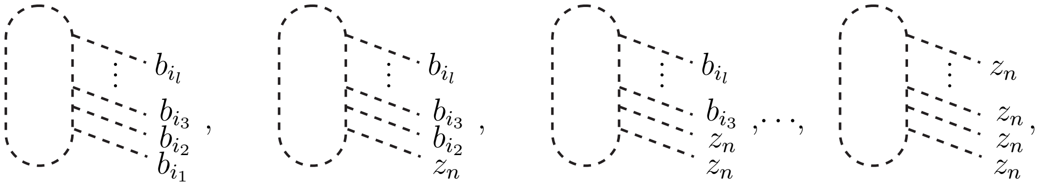
$$W_L(\omega_l)=\sum \mathrm{Trace}([a_{i_1},[a_{i_2},[a_{i_3},\cdots,[a_{i_l},\cdot\,]]]])b_{i_1}\otimes b_{i_2}\otimes b_{i_3}\otimes\cdots\otimes b_{i_l}$$

where the summand corresponds to the following graphical expression, and where a_{i_j} and b_{i_j} are in L.

[1]It is remarked that commutative parts may be 0-dimensional. See the head of Chapter 5.



Let us consider L as $\mathcal{S}_n$, a solvable Lie algebra defined in the previous section. The summands in the lemma are like as



where each b_{i_j} is in $\mathcal{S}_{n-1}$. It is noticed if there is no univalent vertex which is marked by z_n, then the corresponding term is considered as in $W_{\mathcal{S}_{n-1}}$. Namely,

$$\begin{aligned}
&\mathrm{Trace}([a_{i_1},[a_{i_2},[a_{i_3},\cdots,[a_{i_l},\,\cdot\,]]]])b_{i_1}\otimes b_{i_2}\otimes b_{i_3}\otimes\cdots\otimes b_{i_l}\\
&\quad=\ \mathrm{Trace}([a_{i_1},[a_{i_2},[a_{i_3},\cdots,[a_{i_l},\,\cdot\,]_0]_0]_0]_0)b_{i_1}\otimes b_{i_2}\otimes b_{i_3}\otimes\cdots\otimes b_{i_l},
\end{aligned}$$

and we have the following lemma.

Lemma 3.2.2

$$W_{\mathcal{S}_n}(\omega_l)=W_{\mathcal{S}_{n-1}}(\omega_l)+\begin{pmatrix}\text{terms having } z_n\text{'s on}\\ \text{some univalent vertices}\end{pmatrix}.$$

3.2.1 The proof of Theorem 1

Theorem 1 is proved by the following two propositions.

Proposition 3.2.3 *The loop-order of the weight system derived from a nilpotent Lie algebra is* -1.

Proposition 3.2.4 *If the loop-order of* W_L *derived from a Lie algebra* L *is* -1*, then* L *is nilpotent.*

The proof of Proposition 3.2.3. It is elementary. Let L be a nilpotent Lie algebra. Let us consider a Jacobi diagram D of loop-degree ≥ 0 and we can take a loop with some trivalent vertices in D (Figure 3.1) which is isomorphic to a wheel diagram. Since any adjoint map $[a,\,\cdot\,]$ of a nilpotent Lie algebra does not have the semisimple part, the traces in Lemma 3.2.1 vanish. Therefore the contraction is vanished on the whole of D. As a conclusion, if a Jacobi diagram D has a loop, then $W_L(D)$ vanishes. $\square$

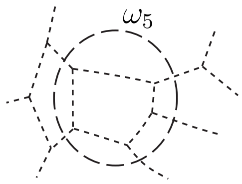


Figure 3.1: A wheel part in the proof of Theorem 1

The proof of Proposition 3.2.4. We will prove the contraposition of the proposition. For non solvable algebras the loop-orders are positive (See Section 3.2.4), therefore we concentrate the solvable cases. Let L' be a solvable Lie algebra and L be an extension of it with a center z and a derivation $\bar{d}$. Assume that $\bar{d}$ is not nilpotent i.e. the algebra L is not nilpotent.

Let us recall the formula in Lemma 3.2.1, and consider the term of it such that all a_{i_k} are equal to $\bar{d}$ (thus b_{i_k} is z). Explicitly, the term is expressed as

$$\mathrm{Trace}([d, [d, \cdots, [d, \cdot\,]]])z^{\otimes l}. \tag{3.1}$$

We can check that it does not vanished for some l as follows. Since the derivation $\bar{d}$ is not nilpotent, we can take eigenvalues λ_1, λ_2, $\cdots$, λ_l of it all of which are not zero and their associated eigenvectors. Hence the trace of (3.1) is written as $\sum_i \lambda_i^l$. If the loop-order of W_L is negative, then the sum $\sum_i \lambda_i^l$ must vanish for all l, and it implies each λ_i is zero. It is a contradiction. Therefore there exists ω_l not vanished by W_L. In other words, if L is not nilpotent, then the loop-order of W_L cannot be negative. □

3.2.2 The proof of Theorem 5

Theorem 5 is a direct consequence of the last subsection. We remark that Theorem 2 and Theorem 3, proved in the sequel, do not depend on the contents of this subsection.

Assume that L' is nilpotent and L is an extension of it with a center and non nilpotent derivation d. We can see that $W_L(\omega_l)$ has only one term

$$W_L(\omega_l) = \mathrm{Trace}([d, [d, \cdots, [d, \cdot\,]]])z^{\otimes l}.$$

Let D be a Jacobi diagram of loop-degree ≥ 1. Then D has a wheel ω_l as a sub diagram as pictured in Figure 3.1. In evaluation with W_L, some brackets with a central element z appear, and whole evaluation $W_L(D)$ vanishes (Figure 3.2).

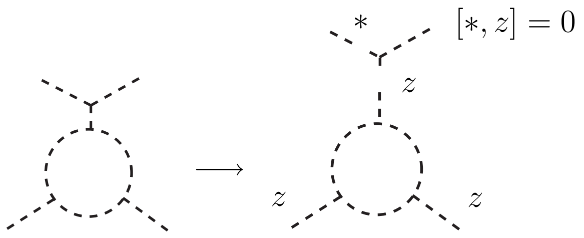


Figure 3.2: The last part of Theorem 5

3.2.3 The proof of Theorem 2

Let L be a Lie algebra. We suppose that a Jacobi diagram D of loop-degree ≥ 1 having l univalent vertices is not vanished by a weight system W_L. We will prove Theorem 2 by constructing another diagram having higher loop-degree and not being vanished by W_L.

In the evaluation $W_L(D)$, a tensor expressed as $D_{i_1 i_2 \cdots i_l} u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_l}$ appear, where the j-th univalent vertex of D is marked by u_{i_j} and where $D_{i_1 i_2 \cdots i_l}$ is the coefficient of it. In other words, $W_L(D) = \sum D_{i_1 i_2 \cdots i_l} u_{i_1} \otimes u_{i_2} \otimes \cdots \otimes u_{i_l}$ where the summation runs all over possible ways of marking univalent vertices. The non-vanishing of $W_L(D)$ implies the existence of non-zero $D_{i_1 i_2 \cdots i_l}$ for some u_{i_j}'s. Explicitly the coefficients are written by repeated brackets and traces,

$$D_{i_1 i_2 \cdots i_l} = \mathrm{Trace}(\sum_k [[u_{i_1}, [u_{i_2}, \cdots, [u_{i_m}, u_k]]], [u_{i_{m+1}}, \cdots, [u_{i_l}, [v_k, \cdot\,]]]]).$$

Let us take an edge of D, such that it connects to two trivalent vertices, and that the diagram obtained by removing the edge still contains a loop (edges connecting to univalent vertices will be moved away with the IHX relation, if necessary). By unplugging the distinguished edge, we can define an endomorphism $\phi_{D'}$ of L, and it is written as

$$\phi_{D'} = \sum_k [[u_{i_1}, [u_{i_2}, \cdots, [u_{i_m}, u_k]]], [u_{i_{m+1}}, \cdots, [u_{i_l}, [v_k, \cdot\,]]]].$$

The non-vanishing of $\mathrm{Trace}(\phi_{D'})$ implies the existence of its eigenvalues $\{\lambda_i\}$ at least one of which is not zero. Therefore, $\mathrm{Trace}(\phi_{D'})^k = \sum \lambda_i^k$ is not zero for some $k > 1$ and it implies that non-vanishing of the trace. The endomorphism $(\phi_{D'})^k$ corresponds to a connected Jacobi diagram D_k which is a chain of D (Figure 3.3) and is linearly independent of the other terms in $W_L(D_k)$. Hence a Jacobi diagram of loop-degree $k > 1$ which is not vanished by W_L is obtained. This completes the proof of Theorem 2.

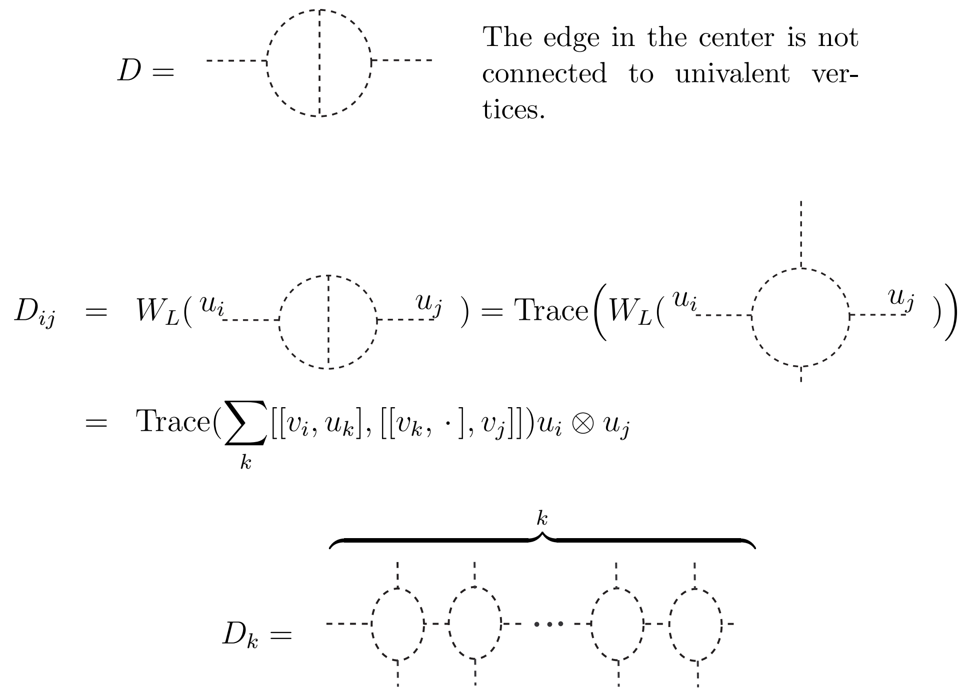


Figure 3.3: The proof of Theorem 2

3.2.4 The proof of Theorem 3

Let L be a Lie algebra. We notice that Theorem 1 implies -1 is only possible loop-order which is negative. Suppose that the loop-order of W_L is neither -1 nor 0, namely it is positive. Then there exist a Jacobi diagram D such that it is not vanished by W_L and its loop-degree is larger than or equal to 1. Due to Theorem 2, we can see the loop-order is ∞.

Theorem 1 and Theorem 5 say existences of weight systems of loop-orders -1 and 0 respectively, and the lemma below assures existences of weight systems of loop-orders ∞. It completes the proof of Theorem 3. $\square$

Lemma 3.2.5 *Let L be a Lie algebra which is not solvable. Then there exists a Jacobi diagram D having a positive loop-degree such that $W_L(D) \neq 0$.*

PROOF. At first we discuss the case $L = \mathfrak{g}$ is semisimple. It is clear that there exist Jacobi diagrams having sufficiently large loop-orders because of the semisimplicity of $\mathfrak{g}$.

For a not solvable Lie algebra L, let us define L' as the quotient of L by its maximal solvable ideal. It is noticed that L' is semisimple. The evaluation $W_L(D')$ for a Jacobi diagram D' has terms whose contractions consist of elements in L'. Let us choose D' which is not vanished by $W_{L'}$ and which has a large loop-degree, and it is not also vanished by W_L. $\square$

Chapter 4

Weight systems and nilpotent Lie algebras

In this chapter, we define a class of nilpotent Lie algebras and research the weight system derived from them. Those algebras are tensor products of metrized Lie algebras and nilpotent associative algebras. The fact, those algebras have invariant metrics, proves Theorem 4 (it is essentially Corollary 4.1.2). We also see that those nilpotent Lie algebras are related to the Milnor invariants of string links (Theorem 7).

4.1 Algebras and weight systems

Let $\mathfrak{g}$ be a semisimple Lie algebra. We can take on it a non-degenerate invariant bilinear form $(\cdot\,|\,\cdot)$, called the Killing form. Let us denote by $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_m\}$ two bases of $\mathfrak{g}$ satisfying $(u_i|v_j) = \delta_{ij}$. Then it is known that an invariant 2-tensor can be written as $\sum(u_i \otimes v_i + v_i \otimes u_i)$.

Let n be a positive integer and X be an indeterminate. We define an associative algebra without the unit as $(\bigoplus_{i\geq 1} \mathbb{C}X^i)/(X^{n+1})$ and denote it by $N(n)$. The tensor product $\mathfrak{g}(n) = \mathfrak{g} \otimes N(n)$ is a Lie algebra[1] with the bracket product

$$[a \otimes X^i, b \otimes X^j] = [a, b] \otimes X^{i+j},$$

where a and b are in $\mathfrak{g}$ and the bracket in the RHS is of $\mathfrak{g}$. One can check easily that $\mathfrak{g}(n)$ is finite-dimensional and is nilpotent. In the rest of this chapter, we write $a \otimes X^i \in N(n)$ as $a^{(i)}$ for short. It is clear that $a^{(k)} = 0$ for $k > n$. The algebra $\mathfrak{g}(n)$ also has a non-degenerate invariant bilinear form $(\cdot\,|\,\cdot)'$ defined by

$$(a^{(i)}|b^{(j)})' = (a|b)\delta_{i+j,n+1}$$

[1]It is found in the book [22].

and hence an invariant 2-tensor exists and is expressed as

$$t = \sum_{1 \leq i \leq n} \sum_{j} u_j^{(i)} \otimes v_j^{(n+1-i)}.$$

Example. Let us take $\mathfrak{g} = sl_2$. It is generated by x, y, and h with relations $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$. An invariant 2-tensor of sl_2 with respect to the Killing form is $\frac{1}{2}h \otimes h + x \otimes y + y \otimes x$. Then, the nilpotent Lie algebra $sl_2(n) = sl_2 \otimes N(n)$ is generated by $h^{(1)}$, $x^{(1)}$ and $y^{(1)}$, and an invariant 2-tensor of $sl_2(n)$ is given by $t = \sum_i (\frac{1}{2}h^{(i)} \otimes h^{(n+1-i)} + x^{(i)} \otimes y^{(n+1-i)} + y^{(i)} \otimes x^{(n+1-i)})$.

We can define the weight system $W_{\mathfrak{g}(n)}$ following Chapter 2 and we evaluate it now. The nilpotentness of $\mathfrak{g}(n)$ implies that its loop-order is -1 and we have the following results.

Let $k > 0$ be an integer and consider a set $S_k = \{(a_1, a_2, \cdots, a_{k+1}) | a_i \in \mathbb{N}, \sum a_i = k(n+1)\}$. Three facts are noticed. For $(s_1, s_2, \cdots, s_{k+1}) \in S_k$ each s_i is greater than $k-1$ and smaller than $n+1$, a tree connected Jacobi diagram of degree k has $k+1$ univalent vertices, and for a metrized Lie algebra L and for a Jacobi diagram D, $W_L(D) \in S(L)$.

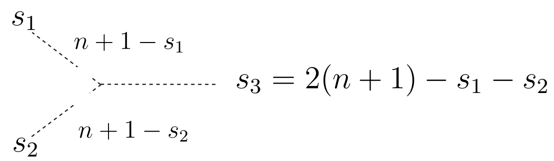
Proposition 4.1.1 *Let D be a connected tree Jacobi diagram of degree k and write down $W_{\mathfrak{g}}(D) = \sum_i g_{i,1} \otimes g_{i,2} \otimes \cdots \otimes g_{i,k+1}$, where each $g_{i,j}$ is in $\mathfrak{g}$. Then,*

$$W_{\mathfrak{g}(n)}(D) = \sum_{S_k} \sum_{i} g_{i,1}^{(s_1)} \otimes g_{i,2}^{(s_2)} \otimes \cdots \otimes g_{i,k+1}^{(s_{k+1})},$$

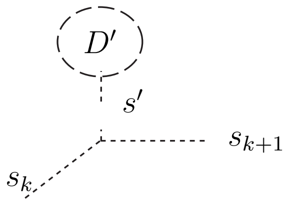
where $(s_1, s_2, \cdots, s_{k+1}) \in S_k$. Moreover, if k is greater than $n-1$, then $W_{\mathfrak{g}(n)}(D) = 0$.

For each term in $W_{\mathfrak{g}(n)}(D)$, $g_{i,j}^{(s_j)}$ is assigned to the j-th univalent vertex. We call s_j the level of the vertex in the term.

Proof. It is observed easily that $W_{\mathfrak{g}(n)}(D)$ is written as a summation of expressions $\sum_i g_{i,1}^{(s_1)} \otimes g_{i,2}^{(s_2)} \otimes \cdots \otimes g_{i,k+1}^{(s_{k+1})}$. We will check that it is summed up over S_k by induction. First, for degree 1, that is D is a single chord, the diagram is mapped to the invariant 2-tensor t. Thus $s_1 + s_2 = n + 1$ (the sum of levels of terms in t) holds. Second, we will check for diagrams of degree 2, it has only one trivalent vertex. The evaluations are as follows, focusing on the levels.



It shows the equality $s_1 + s_2 + s_3 = 2(n+1)$. Assume that $\sum s_i = k(n+1)$ for less than or equal to $k-1$ ($k > 1$). D has a trivalent vertex which is connected to two univalent vertices, and the labels of these univalent vertices are assumed to be k and $k+1$. Unplugging the edge connecting to the trivalent vertex (see the figure below) makes a Jacobi diagram D' of degree $k-1$.



Let s' be the level of the univalent vertex generated by unplugging and we have $\sum_{l=1}^{k-1} s_l + s' = (k-1)(n+1)$ by the assumption. Plugging the edge again determines the value of s_{k+1} as follows

$$s_{k+1} = s' + (n+1-s_k).$$

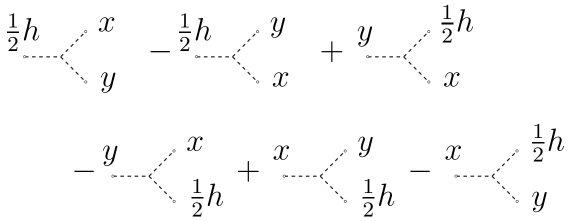
Therefore, the sum of levels in D is $k(n+1)$ by the following calculation.

$$\begin{aligned}\sum_{l=1}^{k+1} s_l &= \sum_{l=1}^{k-1} s_l + s_k + (s' + n + 1 - s_k) \\ &= (k-1)(n+1) + n + 1 \\ &= k(n+1). \quad \square\end{aligned}$$

The following corollary is a direct consequence of the last proposition.

Corollary 4.1.2 *Let D be a tree connected Jacobi diagram in $\mathcal{B}$ of degree $< n$. Then $W_{\mathfrak{g}(n)}(D) \neq 0$.*

Example. Let us consider $sl_2(5)$ defined in the previous example. Let D be a tree Jacobi diagram of degree 2. The evaluation of $W_{sl_2}(D)$ is



and the fact, the set S_2 consists of $(3,1,1)$, $(2,2,1)$ and their permutations, lead us to the calculation of $W_{sl_2(5)}(D)$. It is

$$
\begin{aligned}
W_{sl_2(5)}(D) \;=\; & \tfrac{1}{2}h^{(3)} \begin{array}{l} x^{(1)} \\ y^{(1)} \end{array} - \tfrac{1}{2}h^{(3)} \begin{array}{l} y^{(1)} \\ x^{(1)} \end{array} + y^{(3)} \begin{array}{l} \tfrac{1}{2}h^{(1)} \\ x^{(1)} \end{array} + \cdots \\
& + \tfrac{1}{2}h^{(1)} \begin{array}{l} x^{(1)} \\ y^{(3)} \end{array} - \tfrac{1}{2}h^{(1)} \begin{array}{l} y^{(1)} \\ x^{(3)} \end{array} + y^{(1)} \begin{array}{l} \tfrac{1}{2}h^{(1)} \\ x^{(3)} \end{array} + \cdots \\
& + \tfrac{1}{2}h^{(1)} \begin{array}{l} x^{(3)} \\ y^{(1)} \end{array} - \tfrac{1}{2}h^{(1)} \begin{array}{l} y^{(3)} \\ x^{(1)} \end{array} + y^{(1)} \begin{array}{l} \tfrac{1}{2}h^{(3)} \\ x^{(1)} \end{array} + \cdots \\
& + \cdots \quad .
\end{aligned}
$$

Remark. There are not many nilpotent Lie algebras with metrics [24]. Many nilpotent Lie algebras are realized as graded vector subspaces in Kac-Moody algebras [27], therefore generalized Cartan matrices of them are assigned as invariants of nilpotent Lie algebras. Let fix such a nilpotent Lie algebra. If its generalized Cartan matrix is of finite type, then there exist invariant bilinear forms but they are degenerate. Although we can define a weight system from such an algebra, it vanishes all Jacobi diagrams except for a single chord and a diagram like the letter 'Y' (degree 2 tree diagram). If the generalized Cartan matrix is of indefinite type, then there are a few invariant bilinear forms. Only for affine generalized Cartan matrices there are efficient nilpotent algebras. Algebras defined in this section are examples of such algebras.

4.2 Relations to the Milnor invariant of string links

In [9], Habegger and Masbaum discussed quantum invariants of string links, in particular link-homotopy/concordance invariants.

A string link is an extension of a pure braid. A component of a string link can go both upward and downward locally, and can has maximal/minimal points and self-linkings (Figure 4.1).

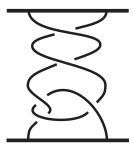


Figure 4.1: A 2-string link

In this section, the supports of Jacobi diagrams are fixed to l segments. Let us recall the space of such Jacobi diagrams are denoted by $\mathcal{A}(l)$.

We define two types of quotients of $\mathcal{A}(l)$. The algebra $\mathcal{A}^t(l)$ is a quotient by the subspace consisting of diagrams with non-simply connected webs. In other words, the subspace is generated by the diagrams with a web of loop-degree > 0.

Remark. We assign the index of the strand to each univalent vertex, on which the vertex stand, and we can forget the supports of Jacobi diagrams by the following reason. If there are two univalent vertices on the same strand, then they can be permuted due to the STU relation. Hence diagrams are independent of the order of univalent vertices on the same strand.

We define the second quotient $\mathcal{A}^h(l)$. Let us consider the subspace generated by diagrams such that at least two univalent vertices in each diagram is attached to the same component of the support. The algebra $\mathcal{A}^h(l)$ is defined as the quotient of $\mathcal{A}^t(l)$ by the subspace. Of course we can forget the strands of diagrams in $\mathcal{A}^h(l)$.

The Kontsevich invariant of string links is a linear map $\hat{Z} : \{l\text{-string links}\} \to \mathcal{A}(l)$. It is naturally defined the linear map $\hat{Z}^t : \{l\text{-string links}\} \to \mathcal{A}^t(l)$. ($\hat{Z}^h$ is also defined similarly). See [9] in order to check the definition of the Milnor invariants of string links.

Theorem 6 ([9, Theorem 6.1]) *Let σ be an l-string link. Assume that $\hat{Z}^t(\sigma) - 1$ does not have terms of degree $< n$, that is*

$$\hat{Z}^t(\sigma) = 1 + \xi + (\text{terms of deg} > n),$$

where ξ is connected tree Jacobi diagram of $\deg = n$. *Then the first non-vanishing Milnor invariants of σ are determined by ξ.*

This theorem says that only tree Jacobi diagrams are needed to calculate the Milnor invariants. Let us recall that weight systems defined in the previous subsection pick up only such diagrams. It is remarked that a weight system w is modified to $\hat{w}$ according to the rule $\hat{w}(D) = w(D)h^{\deg D}$ where h is a parameter.

Theorem 7 *For a string link σ, the value $\hat{W}_{\mathfrak{g}(n)} \circ \hat{Z}(\sigma)$ is a polynomial in parameter h whose degree is at most n. Moreover the lowest terms of it induce the Milnor invariants of σ.*

Proof. It is a direct consequence of Theorem 6 and Proposition 4.1.1. $\square$

Let us remember that a representation corresponds to a strand of string links. Precisely for $D \in \mathcal{A}(l)$ and for a representation (R, V), $W_{\mathfrak{g}(n),R}(D)$ induces an endomorphism of $V^{\otimes l}$.

Let us take the adjoint representation of $\mathfrak{g}(n)$ denoted by ad. In the rest of this section, we show that $W_{\mathfrak{g}(n),\mathrm{ad}}$ realizes the projection $\mathcal{A}(X) \to \mathcal{A}^h(X)/(\deg > n)$.

Let D be a connected tree Jacobi diagram in $\mathcal{A}(l)$ of degree k. The image of $v_1^{(i_1)} \otimes v_2^{(i_2)} \otimes \cdots \otimes v_l^{(i_l)} \in V^{\otimes l} = \mathfrak{g}(n)^{\otimes l}$ mapped by $W_{\mathfrak{g}(n),\mathrm{ad}}(D)$ is a summation of tensors like as $w_1^{(j_1)} \otimes w_2^{(j_2)} \otimes \cdots \otimes w_l^{(j_l)}$ where each $w_i^{(j_i)}$ is in V.

Lemma 4.2.1 *With the notations above, $\sum_p j_p = \sum_p i_p + k(n+1)$ holds for any term in $W_{\mathfrak{g}(n),\mathrm{ad}}(D)(v_1^{(i_1)} \otimes v_2^{(i_2)} \otimes \cdots \otimes v_l^{(i_l)})$.*

PROOF. In adjoint representation, an element $a^{(i)} \in \mathfrak{g}(n)$ raises the level of elements in V by i. With the same notations in Proposition 4.1.1, a univalent vertex in D means the action on V by $g_{i,j}^{(s_j)}$. Hence, the account of the changes of levels are $\sum_j s_j = k(n+1)$. It proves the proposition. □

Proposition 4.2.2 *The weight system $W_{\mathfrak{g}(n),\mathrm{ad}}$ annihilates a diagram such that two univalent vertices in it lie in the same strand.*

PROOF. Assume that the i-th and j-th univalent vertices are connected to a strand c, and that to the vertices two elements $a_i^{(s_i)}$ and $a_j^{(s_j)}$ in $\mathfrak{g}(n)$ are assigned. The following inequality,

$$\begin{aligned} s_i + s_j &\geq k(n+1) - n(k-1) \\ &= k + n \end{aligned}$$

is clear by Lemma 4.2.1. Thus, along the strand c, $v^{(p)} \in V$ is transformed to $(a_i a_j \cdot v)^{(p+k+n)}$, which is vanished in $V = \mathfrak{g}(n)$. □

Chapter 5

Weight systems derived from solvable Lie algebras

The next example is solvable but not nilpotent Lie algebras. The typical one is Heisenberg Lie algebras. It is constructed from the viewpoint of solvable Lie algebras with metrics (Chapter 3) as follows. $\mathcal{S}_1 = \langle z_1, d_1 \rangle$, $\mathcal{S}_i = \mathcal{S}_{i-1} \oplus \langle z_i, d_i \rangle$ with $\bar{d}_i \equiv 0$ $(2 \leq i \leq n)$ and $\mathcal{S}_{n+1} = \mathcal{S}_n \oplus \langle z_{n+1}, d_{n+1} \rangle$ with the derivation $\bar{d}_{n+1}$ defined as

$$\bar{d}_{n+1}(z_i) = z_i, \qquad \bar{d}_{n+1}(d_i) = -d_i$$

for $1 \leq i \leq n$. But we define by another way, from the viewpoint of triangular decompositions.

5.1 Weight systems derived from Heisenberg Lie algebras

Many results in the section are direct consequences of Section 3.2. In addition to them, we show a relations to knot invariants[1].

5.1.1 Heisenberg Lie algebras

The Heisenberg Lie algebra $\mathcal{H}_n$ is generated by $(2n+2)$ symbols $\{x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_n, c, d\}$, and non-vanishing bracket products of them are followings.

$$\begin{aligned} [x_i, y_i] &= c, \\ [d, x_i] &= x_i, \text{ and} \\ [d, y_i] &= -y_i. \end{aligned}$$

[1]The contents of the section is a summary of [23].

The metric of $\mathcal{H}_n$ that we use is defined for generators as follows.

$$(x_i|y_i) = (c|d) = 1, \qquad (c|c) = \lambda$$

and other pairs are zero, where λ is a parameter. Invariant 2-tensors are obtained by taking duals, explicitly,

$$t = \lambda c \otimes c + (c \otimes d + d \otimes c) + \sum_i (x_i \otimes y_i + y_i \otimes x_i).$$

A linear representation of $\mathcal{H}_n$ is infinite-dimensional if it is not trivial. The non-trivial ones are realized as a polynomial ring. Let P_n be a polynomial ring with variables z_1, z_2, …, z_n. On a monomial μ in P_n, generators of $\mathcal{H}_n$ act as

$$\begin{aligned} c \cdot \mu &= c\mu, \\ d \cdot \mu &= -(\deg \mu)\mu, \\ x_i \cdot \mu &= c\frac{\partial}{\partial z_i}\mu, \text{ and} \\ y_i \cdot \mu &= z_i \mu, \end{aligned}$$

where we abuse the symbol c, it also expresses the eigenvalue appearing in the action of the generator c.

5.1.2 Weight systems on webs

Throughout the section, the weight system derived from $\mathcal{H}_n$ is denoted by W_n.

Because of Theorem 5, the loop-order of W_n is 0. The explicit evaluations are followings.

Proposition 5.1.1 *The tensor $W_n(D)$ is non-zero for only three types of Jacobi diagrams.*

1. *A chord:* $W_n(D) = t = \lambda c \otimes c + (c \otimes d + d \otimes c) + \sum_i (x_i \otimes y_i + y_i \otimes x_i)$.
2. deg $= 2$ *tree:* $W_n(D) = -\sum_i \sum_{\sigma \in S_3} (-1)^{\mathrm{sgn}\sigma} \sigma(z \otimes x_i \otimes y_i)$, *where S_3 is the symmetric group of order* 3 *and σ acts on $\mathcal{H}_n^{\otimes 3}$ as a permutation.*
3. *A serial tree, each trivalent vertex in it connects at least one univalent vertex(depicted in Figure 5.1). The value $W_n(D)$ is*

$$\sum_i \{(x_i \otimes z - z \otimes x_i) \otimes z^{\otimes(l-2)} \otimes (z \otimes y_i - y_i \otimes z) \\ +(-1)^l (y_i \otimes z - z \otimes y_i) \otimes z^{\otimes(l-2)} \otimes (z \otimes x_i - x_i \otimes z)\}$$

where l is the degree of D.

4. *A wheel:* $W_n(\omega_k) = \begin{cases} 2nc^{\otimes k} & (k \text{ is even}) \\ 0 & (odd) \end{cases}$

We show in Figure 5.1 that the evaluations. There is a remark. The commutativities and orthogonalities of generators of different indices implies that a non-vanishing term in $W_n(D)$ is contracted with x_i, y_i, c and d where i is fixed (c.f. Lemma 2.2.2). Thus it is sufficient to see with x_1 and y_1.

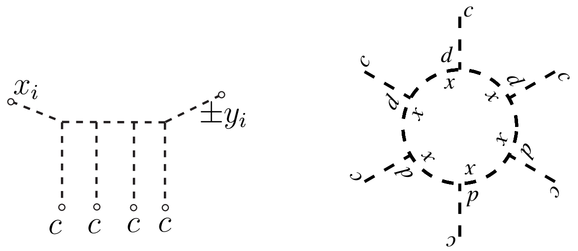


Figure 5.1: Non-vanishing diagrams

5.1.3 Weight systems on supported Jacobi diagrams

Here we discuss the cases the support is a circle, or a single segment, referred an 'arrow'.

If the support of a Jacobi diagram has some circles, then W_n always diverges. It is the reason that it is a trace of a representation (See Lemma 2.2.6), and the representations we use are infinite-dimensional. However some quantities are obtained if the support is a circle. Because it is well known that Jacobi diagrams on a circle and on an arrow are equivalent (See Figure 5.2).

Lemma 5.1.2 ([2]) *Let $\hat{D}$ be a Jacobi diagram on a circle. Let D_1 and D_2 be Jacobi diagrams on an arrow obtained by cutting the support of $\hat{D}$ at distinct points p_1 and p_2 respectively. Then D_1 and D_2 equal in $\mathcal{A}(1)$.*

For $D \in \mathcal{A}(1)$ obtained by cutting the support of $\hat{D} \in \mathcal{A}(S^1)$, a scalar function $W'_n(\hat{D})$ is defined as $W_n(D)$. It is well-defined because of the next lemma. It is derived from the irreducibility of the polynomial representation and Schur's lemma.

Lemma 5.1.3 *The value of the weight system $W_n(D)$ is the scalar multiplication operator $P_n \to P_n$ for $D \in \mathcal{A}(1)$.*

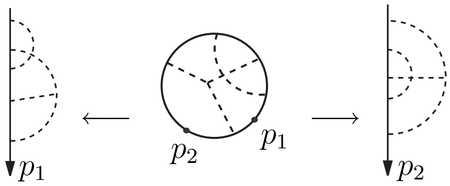
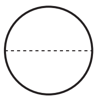
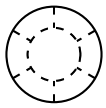
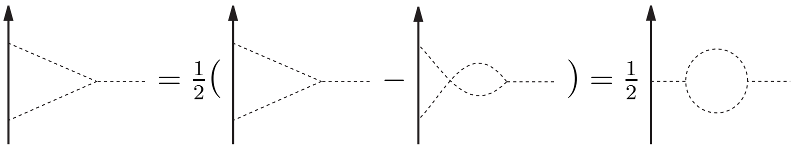


Figure 5.2: An example of Lemma 5.1.2

The next proposition is a specialization of Proposition 5.1.1. The diagram with a single chord on a circle is denoted by Θ, and a wheel ω_l on a circle is done by $\hat{\omega}_l$.

$\Theta =$  $\hat{\omega}_6 =$ 

It is noticed that on $\mathcal{A}(S^1)$ (or $\mathcal{A}(1)$) tree diagrams are transformed to diagrams having loops with STU relation



Proposition 5.1.4 *The values of W'_n on $\mathcal{A}(S^1)$ are as follows.*

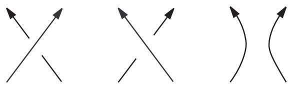
$$W'_n(\hat{D}) = \begin{cases} \lambda c^2 + nc & \text{if } \hat{D} = \Theta, \\ 2nc^{2l} & \text{if } D = \hat{\omega}_{2l}, \\ 0 & \text{otherwise.} \end{cases}$$

The Alexander-Conway polynomial $C(K;h)$ for a framed knot K is a power series in the formal parameter h defined by the following two rules:

$$C(\text{unknot}; h) = 1,$$

$$C(K_+;h) - C(K_-;h) = (e^{\frac{h}{2}} - e^{-\frac{h}{2}})C(K_0;h).$$

Here, three knots K_+, K_- and K_0 are identical except for small balls in them. Inside of the balls are the followings.



K_+ K_- K_0

The Kontsevich invariant maps a knot to a series of Jacobi diagrams on S^1. As in Chapter 4 we modify a weight system w to $\hat{w}$ with the rule $\hat{w}(D) = w(D)h^{\deg D}$.

There is a formula that coefficients of the Alexander-Conway polynomial relate to the Kontsevich invariant of knots. Suppose that $C(K;h)$ for a knot K is expanded as follows.

$$C(K;h) = \exp(-2\sum_{m=1}^{\infty} d_{2m}(K)h^{2m}).$$

Suppose that $C(K;h)$ for a knot K is expanded as follows.

$$C(K;h) = \exp(-2\sum_{m=1}^{\infty} d_{2m}(K)h^{2m}).$$

Theorem 8 ([5]) *The following formula holds.*

$$\hat{Z}(K) = \exp(\frac{f(K)}{2}\Theta + \sum_{m=1}^{\infty} d_{2m}(K)\hat{\omega}_{2m} + X),$$

where f is the framing of K, and X is a sum of Jacobi diagrams of loop-degree > 0.

The product in the exponential map is the connected sum, which corresponds to the product of $(1,1)$-tangles, and that of Jacobi diagrams on an arrow.

By applying $\hat{W}'_n$ to the expression in Theorem 8, we have the following formula, on quantum invariants derived from Heisenberg Lie algebras. It also indicates that the resulting invariants are framing independent if $\lambda = -\frac{n}{c}$.

Theorem 9 *The quantum invariant of knots $\tau_{n,c,\lambda}$ defined by $\hat{W}'_n \circ \hat{Z}$ is the negative power of the Alexander-Conway polynomial. More explicitly, with the same notations above the following equality holds*

$$\tau_{n,c,\lambda}(K) = \exp(\frac{f(K)(\lambda c + n)}{2}ch)\frac{1}{C(K;ch)^n}.$$

5.1.4 Evaluations for horizontal diagrams

In this subsection, all Jacobi diagrams are supposed to be on k oriented segments. They are put vertically, and are oriented to the upper. They are labeled by numbers from the left to the right. The product on $\mathcal{A}(k)$ is defined as the contraction of the tails of intervals with the heads of intervals, and thus $\mathcal{A}(k)$ is an algebra. Of course the Lie bracket $[s_1, s_2]$ (s_1, $s_2 \in \mathcal{A}(k)$) is defined as the difference $s_1s_2 - s_2s_1$. Moreover this (bracket) product is compatible with weight systems dealt in this paper. Precisely, for a weight system w and s_1, $s_2 \in \mathcal{A}(k)$, $w(s_1s_2) = w(s_1)w(s_2)$.

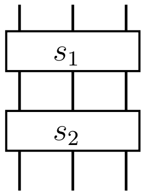
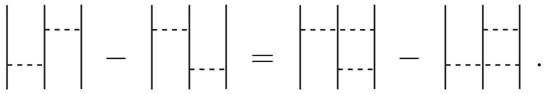


Figure 5.3: The product $s_1 s_2$

In this subsection, the image by W_1 on $\mathcal{A}(k)$ are discussed, especially on $\mathcal{A}(k)$ as a Lie algebra. We restrict Jacobi diagrams so that all webs are horizontal intervals, and the relations of Jacobi diagrams are reduced to the 4-term relation presented as follows (and their permutations)



Let us fix k to be 3, and denote elementary Jacobi diagrams in $\mathcal{A}(3)$ as in Figure 5.4 by T_1 and T_2, respectively.

Figure 5.4: Jacobi diagrams T_1 and T_2.

Lemma 5.1.5 *The weight system W_1 maps a Lie algebra $\mathcal{A}(3)$ to a three-dimensional free module over the polynomial ring generated by $c_1 = c \otimes 1 \otimes 1$, $c_2 = 1 \otimes c \otimes 1$ and $c_3 = 1 \otimes 1 \otimes c$. Generators of the module are a tensor $Q = [t \otimes 1, 1 \otimes t]$ and $[W_1(T_i), Q]$ ($i = 0$, 1).*

The explicit expression of Q is as follows.

$$\begin{aligned} Q &= [t \otimes 1, 1 \otimes t] \\ &= c \otimes x \otimes y - c \otimes y \otimes x - x \otimes c \otimes y \\ &\quad + x \otimes y \otimes c + y \otimes c \otimes x - y \otimes x \otimes c. \end{aligned}$$

The next is the case $k = 4$. The commutators of the permutations of Q's are simple in the case. They are obtained by direct calculations and by reflections. We use the following convention. Let $A = a_1 \otimes a_2 \otimes a_3$ be a tensor. A_{ijk} is a tensor $a'_1 \otimes a'_2 \otimes a'_3 \otimes a'_4$ determined by the rule $a'_i = a_1$, $a'_j = a_2$, $a'_k = a_3$, and the rest is 1. It is remarked that the notations c_1, c_2, and c_3 defined above are compatible with these conventions and that the reflection Q_{321} equals $-Q$.

Lemma 5.1.6 *The commutators of Q_{ijk}'s are the following.*

$$\begin{aligned}
[Q_{234}, Q_{123}] &= c_2c_3(Q_{134} + Q_{124}), \\
[Q_{234}, Q_{134}] &= -c_3c_4(Q_{124} + Q_{123}), \\
[Q_{234}, Q_{124}] &= c_2c_4(Q_{134} - Q_{123}), \\
[Q_{134}, Q_{123}] &= c_1c_3(Q_{124} - Q_{234}), \\
[Q_{134}, Q_{124}] &= -c_1c_4(Q_{123} + Q_{234}), \quad \textit{and} \\
[Q_{124}, Q_{123}] &= -c_1c_2(Q_{234} + Q_{134}).
\end{aligned}$$

We can calculate the image of an associator by above formulas. Invertible elements Φ in $\mathcal{A}(3)$ and R in $\mathcal{A}(2)$ are called an associator and an R-matrix respectively if they satisfy the following three equalities,

$$\begin{array}{cl}
\Delta_3(\Phi^{-1})\Delta_0(\Phi)\Delta_2(\Phi)\Delta_4(\Phi)\Delta_1(\Phi^{-1}) = 1 & \text{(Pentagon)}, \\
\Delta_1(R) = \Phi_{312}R_{13}(\Phi^{-1})_{132}R_{23}\Phi & \text{(Hexagon)}, \\
\Phi_{321} = \Phi^{-1} & \text{(Unitarity)}.
\end{array}$$

By Drinfeld[2] [8] there is an associator such that $\varphi = \log \Phi$ can be a power series each term of which is a nested bracket product $c_\nu[a_1, [a_2, [a_3, \cdots, [a_{2k-1}, a_{2k}] \cdots]]]$, where each a_i is either of two generators T_1 and T_2, and c_ν is a coefficient.

Let us define Φ_W as $\hat{W}_1(\Phi)$ and define three polynomials,

$$\begin{aligned}
\alpha &= -(c_1c_2 + c_2c_3 - c_1c_3 + c_2^2), \\
\beta_1 &= (c_1 + c_2)^2, \quad \text{and} \\
\beta_2 &= (c_3 + c_2)^2.
\end{aligned}$$

By the fact that φ consists of terms of even degrees and by Lemma 5.1.5, the coefficients of $\hat{W}(\varphi)$ can be regarded as a power series in h and the coefficients are polynomials in α, β_1 and β_2.

Proposition 5.1.7

$$\Phi_W = \exp[(\sum_k \varphi_{2k} h^{2k}) Q h^2],$$

[2] He has not researched on the level of Jacobi diagrams, but on the level of quantized enveloping algebras of semisimple Lie algebras.

where

$$\varphi_{2l} = \sum_{i\geq 0, j\geq k\geq 0, i+j+k=l} a_{i,jk}\alpha^i\beta_1^j\beta_2^k,$$

with the conditions $a_{i,jk} = a_{i,kj}$.

By a computation with *Maple*, we can see that the coefficients $a_{i,jk}$ $(i+j+k < 3)$are given by

$$\begin{aligned}
\varphi(h) \ &= \ a - \Big\{\frac{1}{10}\alpha + \frac{2}{5}(\beta_1+\beta_2)\Big\}a^2h^2 \\
&+\Big\{\frac{2}{105}\alpha^2 + \frac{6}{35}\alpha(\beta_1+\beta_2) + \frac{8}{35}({\beta_1}^2+{\beta_2}^2) + \frac{13}{42}\beta_1\beta_2\Big\}a^3h^4 \\
&+\Big\{-\frac{3}{700}\alpha^3 - \frac{12}{175}\alpha^2(\beta_1+\beta_2) - \frac{6}{35}\alpha(\beta_1^2+\beta_2^2) \\
&\quad -\frac{493}{1400}\alpha\beta_1\beta_2 - \frac{24}{175}(\beta_1^3+\beta_2^3) - \frac{7}{25}(\beta_1^2\beta_2+\beta_1\beta_2^2)\Big\}a^4h^6 \\
&+(\text{terms of degree} \geq 8)
\end{aligned}$$

where $a = a_{0,00} = -\frac{1}{24}$.

Remark. The computation of Φ_W is achieved by Lieberum [18]. He has decided the equation with the unique solution φ, and present the solution explicitly. Although he has started with the Lie superalgebra $gl(1|1)$, the derived weight system is similar to W_1, especially they coincide for horizontal diagrams. To compute Φ_W, he has not used the Pentagon relation but the trivalent graph which is the skeleton of the tetrahedron.

5.2 Other examples

We calculate weight systems derived from solvable Lie algebras $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ defined in Chapter 3. From $\mathcal{S}_1$ and $\mathcal{S}_2$ weight systems of loop-order at most 0 are obtained. $W_{\mathcal{S}_3}$ is an example of loop-order ∞, which comes from non semisimple Lie algebras.

Let V be an n-dimensional vector space with metric $(\cdot|\cdot)$ and $\{u_i\}$ and $\{v_j\}$ be two bases of V satisfying $(u_i|v_j) = \delta_{ij}$. The translation of the two bases are written by a square matrix T. Let ϕ be an endomorphism of V satisfying $(\phi(x)|y)+(x|\phi(y)) = 0$.

Lemma 5.2.1 *The matrix A presenting ϕ has the property,*

$$T^tA + AT = 0.$$

As in Chapter 3, $\mathcal{C}^{(0)}$ is a commutative Lie algebra generated by $c_i^{(0)}$ $(1 \leq i \leq k_0)$, and $\mathcal{S}_1$ is an extension of $\mathcal{C}^{(0)}$ with two generators z_1 and d_1. Due to the lemma above and the fact that the translation T is a identity matrix, the associated derivation $\bar{d}_1$ is presented by an alternating matrix of size k_0.

The property of $W_{\mathcal{S}_1}$ is clear. $\mathcal{S}_1$ is an extension of a nilpotent Lie algebra and Theorem 5 says that such an algebra derives a weight system of loop-order at most 0.

Let us consider another commutative algebra $\mathcal{C}^{(1)}$ and construct $\mathcal{S}_2$ with $\mathcal{S}_1$ and the commutative algebra. We assume that $\mathrm{Ker}[d_1, \cdot]$ is trivial, especially ad_{d_1} is not nilpotent on $\mathcal{S}_1$. The translation of dual basis is presented by a matrix as follows.

$$\begin{pmatrix} I_{k_0} & O & O \\ O & \sigma & O \\ O & O & I_{k_1} \end{pmatrix}$$

Here I_k means an identity matrix and $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence $\bar{d}_2$ is presented by

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ -\sigma^t A_{12} & A_{22} & A_{23} \\ -{}^t A_{13} & {}^t(\sigma A_{23}) & A_{33} \end{pmatrix},$$

where A_{11} and A_{33} are alternating matrices and A_{22} satisfies $A_{22}\sigma + {}^t(A_{22}\sigma) = 0$. Moreover, the fact that the central elements are mapped into the center by the derivation d_2 implies that $A_{22} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ for some scalar a and that A_{13} and the second column of A_{12} (corresponding to the generator z_1) are zero.

Now we will evaluate $W_{\mathcal{S}_2}$. First notice that $W_{\mathcal{S}_2}(\omega_l)$ has the part $W_{\mathcal{S}_1}(\omega_l)$ (c.f. Formula (3.2.2)). The results are the sum of tensors like as $z_{i_1} \otimes z_{i_2} \otimes \cdots \otimes z_{i_l}$ where $i_j \in \{1, 2\}$. Let D be a Jacobi diagram depicted in Figure 5.5, where the ordering univalent vertices in it is indicated by the numbers. The evaluation of the wheel part is $\mathrm{Trace}(d_{i_1} d_{i_2} d_{i_3}) z_{i_1} \otimes z_{i_2} \otimes z_{i_3}$. Since z_1 is not commutative in $\mathcal{S}_2$ with d_2 only, a univalent vertex labelled by z_1 can be inserted in another vertex. However, $W_{\mathcal{S}_2}$ is of loop-order 0. Finally,

$$\begin{aligned} W_{\mathcal{S}_2}(D) = & (\mathrm{Trace}(d_1^3)) z_1 \otimes z_2 \otimes z_2 \otimes z_1 \\ & + (\mathrm{Trace}(d_1^2 d_2))(z_2 \otimes z_2 \otimes z_2 \otimes z_1 + z_1 \otimes z_2 \otimes z_2 \otimes z_2) \\ & + (\mathrm{Trace}(d_1 d_2^2)) z_2 \otimes z_2 \otimes z_2 \otimes z_2. \end{aligned}$$

A branched wheel, like as depicted in the same figure, appears as a diagram in $\mathcal{A}(S^{\sqcup n})$ or $\mathcal{A}(n)$ except for[3] $n = 1$.

A similar procedure makes an algebra $\mathcal{S}_3$. In this case, z_1 is no longer commutative with $c_i^{(0)}$'s and $W_{\mathcal{S}_3}$ picks a Jacobi diagram with some loops.

[3]For $n = 1$, branched wheels do not appear. See Proposition 5.1.4.

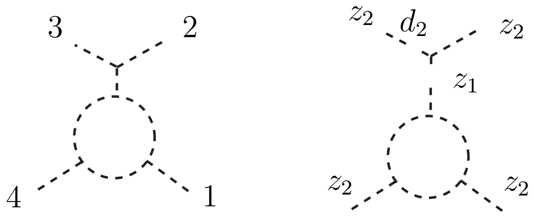


Figure 5.5: Non-vanishing of $W_{\mathcal{S}_2}$

Chapter 6

Problems

Weight systems having positive loop-orders. It is still a problem to construct a weight system of loop-order ≥ 1 and $\leq \infty$. The proof of Theorem 2 seems generic, indeed the theorem can be applied to weight systems defined as contracted tensors. We should take other ways to find weight systems having positive and finite loop-orders, they may be defined with global properties of Jacobi diagrams, and with coefficients which are not complex numbers, etc.
The meaning of loop-degrees from representations. Due to Theorem 3 loop-orders are classified to three cases. What is the reason? The three possibilities -1, 0, and ∞ seems to correspond the degrees of series $(1+x)^1$, $(1+x)^0$ and $(1+x)^{-1}$.
Constructions of string link invariants by another way. As described in Chapter 1, quantum invariants can be defined in two ways. One of them is by Turaev, the definition with quantum groups and another is by Bar-Natan and Kontsevich, defined with the Kontsevich invariant and (ordinary) Lie algebras. The definition with the Kontsevich invariant is suitable to see how topologies and geometries of knot-like objects (they are knots, links, braids, string links, 3-manifolds, etc.) appear in the whole of quantum invariants([] and the present paper). On the other hand, the definition with quantum groups is suitable to calculate invariants of concrete knot-like objects.

In Chapter 4 of this paper, we show the nilpotent Lie algebra $\mathfrak{g}(n)$ corresponds to the Milnor invariants of string links, by use of the Kontsevich invariant. If we translate it to a quantum invariant defined with a quantum group, then we have formulas to calculate the Milnor invariants easily.
Weight systems derived from solvable Lie algebras. Nilpotencies of Lie algebras are related to loop-order -1 (Theorem 1). However, detailed investigations on weight systems derived from solvable Lie algebras are not done. In particular, the determinations of kernels of those weight systems are left.
Classification of weight systems by that of Lie algebras There is a map $\{\text{metrized Lie algebras}\} \to \{\text{weight systems}\}$ Is it injective or surjective? Are there

partial results?

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