An Invariant of 3-manifolds with value in F_5

Hideaki NISHIHARA

1 Background and Motivation

The Kontsevich invariant was defined as an invariant of oriented unframed links first in the paper of Kontsevich [Ko], and was re-defined as an invariant of oriented framed links by Le and Murakami [LM]. It valued in the graded algebra $\mathcal{A}$ of chord diagrams over Q.

It is known that an oriented framed link presents a three-dimensional closed manifold [Li] and that two links present the same three-dimensional manifold if one of them can be transformed to another by the 'Kirby moves'. Therefore if the value of an invariant of links is not changed under the Kirby move, then this invariant is also an invariant of three-dimensional manifolds.

In the paper [LMMO], Le, Murakami, Murakami and Ohtsuki modified the Kontsevich invariant of framed version, and defined an invariant Λ of 3-manifolds. Precisely, it is an image of projection of the algebra of chord diagrams to its quotient $\mathcal{A}_2$ by a new relation P_2 corresponding to the second Kirby move, and is modified by the effect of the first Kirby move, explicitly:

$$\Lambda(M_L) = [\check{Z}(L)][\check{Z}(\ \bigcirc\!\!\!\!\bigcirc\ )]^{-\sigma+}[\check{Z}(\ \bigcirc\!\!\!\!\bigcirc\ )]^{-\sigma-}$$

where $\check{Z}$ means the modified Kontsevich invariant, L is a framed link which presents the manifold M_L, and $\sigma+$(resp. $\sigma-$) is the number of positive(resp. negative) eigenvalues of linking matrix of L.

The structure of $\mathcal{A}_2$ as associative algebra is as follows. $\mathcal{A}_2$ is generated by unit element 1 (corresponding to empty diagram) and a symbol Θ (corresponding to the web diagram in Figure 1), and the product $\Theta \cdot \Theta$ vanishes since $\frac{5}{2}\Theta \cdot \Theta$ is eqaul to zero up to the relation P_2. Therefore $\mathcal{A}_2$ is two-dimentional as linear space over Q spanned by 1 and Θ. The invariant Λ can be described with two invariants with value in Q, Λ_0 and Λ_1, as $\Lambda = \Lambda_0 + \Lambda_1\Theta$. The invariant Λ_0 is determined by the first homology group of M_L, and Λ_1 is proportional to the Casson-Lescop-Walker invariant.

The Kontsevich invariant of links is presented as an infinite series in $\mathcal{A}$, and we obtain the invariant of three-dimensional manifolds, presented by a polynomial of Θ of degree 1, by means of projection from $\mathcal{A}$ to $\mathcal{A}_2$. It seems to be wasteful for the author to cut off terms of higher order in the Kontsevich invariant, and he tried constructing an invariant

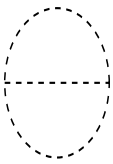


Figure 1: The web diagram denoted by Θ

of three-manifolds valued in the algebra $\mathcal{A}_2'$ whose generators and relations are same as $\mathcal{A}_2$ and whose coefficient ring is F_5, the prime field of characterisitic five, to avoid the relation $\Theta \cdot \Theta = 0$.

2 The algebra of web diagrams

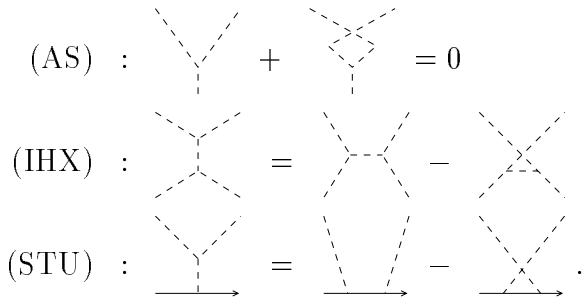
We define web diagrams as trivalent graphs on circles. They are defined and studied in Bar-Natan's paper [BN1] (Web diagrams is called 'chord diagram' in the paper). We can consider the linear space spanned by all web diagrams as graded algebra and Kontsevich invariant of links is obtained as an infinite series in

Let X be a disjoint union of m copies of S^1 's(m may be zero, X is empty set in that case), and let G be a finite graph of which each vertex is univalent or trivalent. Around a trivalent vertex v of G, three edges connected with v are ordered. A chord diagram D with support X is a pair of X (drawn as solid line, called 'string') and G (drawn as dashed line, called 'chord'), and every univalent vertex of G is connected to a point of X.

We denote by $\mathcal{A}(X)$ a certain quotient of the linear space over $\mathbb{Q}$ spanned by all web diagrams with support X:

$$\mathcal{A}(X) = \mathrm{Span}(\text{web diagram on } X)/(\mathrm{STU}, \mathrm{IHX}, \mathrm{AS}),$$

where STU, IHX, and AS mean local relations of web diagrams as follows:



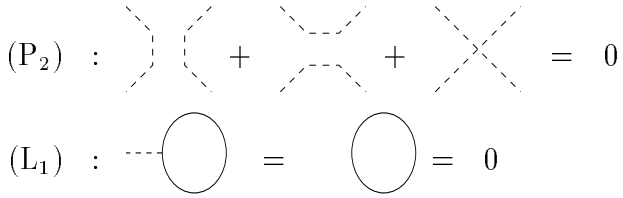
.

We define a degree of a web diagram as half of the number of vertices of it, and $\mathcal{A}(X)$ is graded by the degree. We denote by the same symbol $\mathcal{A}(X)$ the completion of $\mathcal{A}(X)$ by its degree.

3 An invariant of 3-manifolds

In this section, we define the algebra $\mathcal{A}'_2(X)$ and determine the structure of it, and we construct an invariant of 3-manifolds.

First, additional local relations P_2 and L_1 are defined as below, they correspond to the second Kirby move:



The linear space $\mathcal{A}'_2(X)$ over F_5 is defined as follows:

$$\mathcal{A}'_2(X) = \mathrm{Span}(\text{web diagram on } X)/(\mathrm{IHX}, \mathrm{AS}, \mathrm{STU}, \mathrm{P}_2, \mathrm{L}_1).$$

It is noted that another algebra $\mathcal{A}_2(X)$ is definend by the same generators and relations of $\mathcal{A}'_2(X)$, but it is over Q. $\mathcal{A}_2(X)$ is a quotient of $\mathcal{A}(X)$ and the invariant Λ of 3-manifolds, defined in the paper [LMMO], takes value in this algebra.

We can show the following proposition with respect to the algebra $\mathcal{A}'_2(X)$, by splitting a web diagram of higher degree into two web diagrams both of which has at least two vertices by means of relations IHX, L_1, and P_2.

PROPOSITION. *Let D be a web diagram of degree k with support $(S^1)^{\sqcup l}$. If $k \geq l$, then D is equivalent to a web diagram of the form:*

$$c \cdot \bigcirc^{\sqcup k-l} \underbrace{\bigcirc\ \bigcirc \cdots \bigcirc}_{l},$$

where c is a scalar and $\bigcirc^{\sqcup(k-l)}$ means the disjoint union of $k-l$ copies of the web diagram $\bigcirc$. If $k < l$, then D is equivalent to zero.

It is remarked that the proposition above is also true for the algebra $\mathcal{A}_2(X)$. We can remove connected components including solid line from a web diagram of the form below, therefore we can remove that from any web diagram due to the proposition:

$$\bigcirc^{\sqcup k}\ \bigcirc\ \bigcirc \cdots \bigcirc,$$

This operation of web diagrams is considered to be the projection from $\mathcal{A}_2'(X)$(resp. $\mathcal{A}_2(X)$) to $\mathcal{A}_2'(\emptyset)$(resp. $\mathcal{A}_2(\emptyset)$).

The multiplication of $\mathcal{A}_2'(\emptyset)$ is defined as the disjoint union of two web diagrams, and that of $\mathcal{A}_2(\emptyset)$ is defined respectively.

Therefore, we can see the algebraic structure of $\mathcal{A}_2'(X)$. We have to remeber that the symbol Θ represents the distinguished web diagram .

COROLLARY. *The algebra $\mathcal{A}_2'((S^1)^{\sqcup l})$ is isomorphic to the algebra of formal power series $F_5[[\Theta]]$, independently of l.*

We can use the same method of [LMMO] to construct the invariant of 3-manifolds from the Kontsevich invariant of links.

THEOREM. *Let M_L be the three-dimensional manifold presented by framed l-components link L. Then,*

$$\tilde{\Lambda}(M_L) = [\check{Z}(L)][\check{Z}(\ \bigcirc\!\!\!\!\bigcirc\ )]^{-\sigma+}[\check{Z}(\ \bigcirc\!\!\!\!\bigcirc\ )]^{-\sigma-}$$

is invariant under the Kirby moves, therefore it is a topological invariant. In the above statement, $\check{Z}$ means the modified Kontsevich invariant, $\sigma+$(resp. $\sigma-$) does the number of positive(resp. negative) eigenvalues of linking matrix of L, and the operater $[\cdot]$ cuts off the part of web diagrams other than terms of degree l, $l+1$, and $l+2$.

We obtain an invariant of quadratic form with coefficient in F_5. The part of degree zero of $\tilde{\Lambda}(M)$ is congruent to $\Lambda_0(M)$ modulo 5, and that of degree one is also congruent to $\Lambda_1(M)$. Since the part of degree two is independent of terms of degree zero and one (shown in the next section), it may be a new invariant.

REMARK. In the main theorem we cut off the higher degree part of the Kontsevich invariant. It is the reason that denominators of those terms are the multiple of 5. It comes from the fact that the contribution from a crossing in a link to the Kontsevich invariant is exponential. The coefficient of degree 5 part is $1/(5!\cdot 2^5)$. Therefore, for l-component link L the coefficient of the $(l+4)$-th degree part of the Kontsevich invariant of L is the multiple of $1/5$. There is another reason. The Drinfeld associator appearing in the Kontsevich invariant is of the form([BN2]):

$$\Phi_0 + \frac{1}{24}\Phi_2 + \frac{1}{11520}\Phi_4 + \cdots$$

where every Φ_{2n} is an integral linear combination of web diagrams of degree $2n$. Therefore the coefficient of the $(l+3)$-th degree part of the Kontsevich invariant is the multiple of $1/5$.

4 Calculations

Let $K_{p,f}$ be the $(2,p)$-torus knot with framing f (f is positive). We denote by $M_{p,f}$ the closed manifold presented by $K_{p,f}$. The calculation of $\tilde{\Lambda}$ for $M_{p,f}$ results in the form:

$$\tilde{\Lambda}(M_{p,f}) = f + (p^2 + f - 2f^2)\Theta + (-f - 2f^2 + f^3 + 2p + p^2 + fp^2 - 2p^3)\Theta^2$$

There is a remark that the manifold obtained by a connected sum of two manifolds presented by links L_1 and L_2 is presented by a split union of L_1 and L_2, therefore, the value $\tilde{\Lambda}(M_1 \sharp M_2)$ is equal to the product $\tilde{\Lambda}(M_1)\tilde{\Lambda}(M_2)$ cut off parts of degree ≥ 3.

Now we have following calculations:

$$\begin{aligned} \tilde{\Lambda}(M_{10l+3,1}) &= 1 - 2\Theta - 2\Theta^2 \\ \tilde{\Lambda}(M_{8l+3} \sharp M_{6l+1,1}) &= 1 - 2\Theta + (-l^4 + l^2 + l - 2)\Theta^2 \end{aligned}$$

which shows that the term of degree 2 in $\tilde{\Lambda}$ is inependent of the other terms in it.

References

[BN1] D.Bar-Natan, *On Vassiliev knot invariant*, Topology **34**(1995), 423-472

[BN2] D.Bar-Natan, *Non-associative tangles*, Studies Adv. Math. **2** (1997)

[Ki] R.C.Kirby, *A calculus for framed inks in* S^3, Invent. Math.**45**(1978), 35-56

[Ko] M.Kontsevich, *Vassiliev's Knot Invariants*, Adv. in Soviet Math. **16**(1993), 137-150

[LM] T.T.Q.Le and J.Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Composite Math. **102**(1996), 41-64

[LMMO] T.T.Q.Le, H.Murakami, J.Murakami and T.Ohotsuki, *A three-manifold invariant via the Kontsevich integral*, to appear in Osaka J. Math.

[Li] W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*,Ann. Math. **76**(1962)531-540

Department of Mathematics, Graduate School of Science,
Osaka University, 1-1 Machikaneyama-cho, Toyonaka,
Osaka 560-0043, Japan
E-mail address: smv010nh@ex.ecip.osaka-u.ac.jp